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The flag curvature of invariant (α, β)-metrics of type $\frac{(\alpha+\beta)^2}{\alpha}$

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Abstract
In this paper, we study the flag curvature of invariant (α, β)-metrics of the form $\frac{(\alpha+\beta)^2}{\alpha}$ on homogeneous spaces and Lie groups. We give a formula for the flag curvature of invariant metrics of the form $F = \frac{(\alpha+\beta)^2}{\alpha}$ such that α is induced by an invariant Riemannian metric g on the homogeneous space and the Chern connection of F coincides to the Levi-Civita connection of g. Then some conclusions in the cases of naturally reductive homogeneous spaces and Lie groups are given.

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1. Introduction

Finsler geometry is an interesting field in differential geometry which has found many applications in physics and biology (see [1, 2]). One of the important quantities which can be used for characterizing Finsler spaces is flag curvature. The computation of flag curvature, which is a generalization of the concept of sectional curvature in Riemannian geometry, is very difficult. Therefore finding an explicit formula for computing it can be useful for characterizing Finsler spaces. Also it can help us find new examples of spaces with some curvature properties. Working on a general Finsler space for finding an explicit formula for the flag curvature is very computational because of computation in local coordinates. A family of spaces which has many applications in physics is homogeneous spaces (in particular, Lie groups) equipped with invariant metrics. The study of homogeneous spaces (Lie groups) with invariant Riemannian metrics has been a very hot field in recent decades (for example see [8, 13–16]). During recent years, some of these results extended to Finsler spaces in some special cases (see [9–12, 17]). (α, β)-metrics are interesting Finsler metrics which have been studied by many Finsler geometers. Physicists are also interested in these metrics. They seek for some non-Riemannian models for spacetime. For example, by using (α, β)-metrics, G S Asanov introduced Finsleroid–Finsler spaces and formulated pseudo-Finsleroid gravitational field equations (see [3–6]). $F = \frac{(\alpha+\beta)^2}{\alpha}$ is a special (α, β)-metric which has been studied by Shen and Yildirim (see [19]). In this paper, we study flag curvature of these metrics on homogeneous spaces G/H which are invariant under the action G. We suppose that α is induced by an invariant Riemannian metric g on the homogeneous space and the Chern connection of F coincides to the Levi-Civita connection of g. Also we study the special cases when $(G/H, g)$ is naturally reductive or when H is trivial ($H = \{e\}$).

2. Preliminaries

Let M be a smooth manifold. Suppose that g and b are a Riemannian metric and a 1-form on M respectively as follows:

$$g = g_{ij}\mathrm{d}x^i \otimes \mathrm{d}x^j \qquad b = b_i\,\mathrm{d}x^i.$$

In this case we can define a function on TM as follows:

$$F(x, y) = \frac{(\alpha(x, y) + \beta(x, y))^2}{\alpha(x, y)},$$

where $\alpha(x, y) = \sqrt{g_{ij}(x)y^i y^j}$ and $\beta(x, y) = b_i(x)y^i$.

It has been shown F is a Finsler metric if and only if for any $x \in M$, $\|\beta_x\|_\alpha < 1$, where

$$\|\beta_x\|_\alpha = \sqrt{b_i b^i} = \sqrt{g^{ij}b_i b_j}.$$

In a natural way, the Riemannian metric g induces an inner product on any cotangent space T_x^*M such that $\langle \mathrm{d}x^i(x), \mathrm{d}x^j(x)\rangle = g^{ij}(x)$. The induced inner product on T_x^*M induce a linear isomorphism between T_x^*M and T_xM. Then the 1-form b corresponds to a vector field $\tilde{X}$ on M such that

$$g(y, \tilde{X}(x)) = \beta(x, y).$$

Also we have $\|\beta(x)\|_\alpha = \|\tilde{X}(x)\|_\alpha$ (for more details see [12]).

Therefore we can write the Finsler metric $F = \frac{(\alpha+\beta)^2}{\alpha}$ as follows:

$$F(x, y) = \frac{(\alpha(x, y) + g(\tilde{X}(x), y))^2}{\alpha(x, y)},$$

where for any $x \in M$, $\sqrt{g(\tilde{X}(x), \tilde{X}(x))} = \|\tilde{X}(x)\|_\alpha < 1$.

In this paper we use this representation of F (for more details about Finsler metrics see [7, 18]).

3. Flag curvature of invariant metrics of type $\frac{(\alpha+\beta)^2}{\alpha}$ on homogeneous spaces

In this section, we give an explicit formula for the flag curvature of invariant (α, β)-metrics of type $\frac{(\alpha+\beta)^2}{\alpha}$, where α is induced by an invariant Riemannian metric g on the homogeneous space and the Chern connection of F coincides to the Levi-Civita connection of g. For this purpose we need Püttmann's formula for the curvature tensor of invariant Riemannian metrics on homogeneous spaces (see [16]).

Let G be a compact Lie group, H a closed subgroup and g_0 a bi-invariant Riemannian metric on G. Assume that $\mathfrak{g}$ and $\mathfrak{h}$ are the Lie algebras of G and H respectively. The tangent space of the homogeneous space G/H is given by the orthogonal complement $\mathfrak{m}$ of $\mathfrak{h}$ in $\mathfrak{g}$ with respect to g_0. Each invariant metric g on G/H is determined by its restriction to $\mathfrak{m}$. We extend this Ad_H-invariant inner product on $\mathfrak{m}$ to an Ad_H-invariant inner product on $\mathfrak{g}$ by taking g_0 for the components in $\mathfrak{h}$. In this way the metric on G/H determines a unique left invariant, H-biinvariant metric g on G that we also denote by g (see [16]). Suppose that g is an invariant metric on G/H. The values of g_0 and g at the identity are inner products on $\mathfrak{g}$ which we denote as $\langle .,. \rangle_0$ and $\langle .,. \rangle$. The inner product $\langle .,. \rangle$ determines a positive definite endomorphism ϕ of $\mathfrak{g}$ such that $\langle X, Y\rangle = \langle \phi X, Y\rangle_0$ for all $X, Y \in \mathfrak{g}$.

Theorem 3.1. *Let $G, H, \mathfrak{g}, \mathfrak{h}, g, g_0$ and ϕ be as above. Assume that $\tilde{X}$ is an invariant vector field on G/H which is $g(\tilde{X},\tilde{X}) < 1$ and $X := \tilde{X}_H$. Suppose that $F = \frac{(\alpha+\beta)^2}{\alpha}$ is the Finsler metric arising from g and $\tilde{X}$ such that its Chern connection coincides to the Levi-Civita connection of g. Suppose that (P, Y) is a flag in $T_H(G/H)$ such that $\{Y, U\}$ is an orthonormal basis of P with respect to $\langle .,. \rangle$. Then the flag curvature of the flag (P, Y) in $T_H(G/H)$ is given by*

$$K(P,Y) = \frac{6\langle X, R(U,Y)Y\rangle.\langle X,U\rangle + \langle R(U,Y)Y,U\rangle(1-\langle X,Y\rangle^2)}{(1+\langle X,Y\rangle)^4(2\langle X,U\rangle^2 - \langle X,Y\rangle^2+1)}, \tag{3.1}$$

where

$$\begin{aligned}\langle X, R(U,Y)Y\rangle &= \tfrac{1}{4}(\langle[\phi U,Y]+[U,\phi Y],[Y,X]\rangle_0 + \langle[U,Y],[\phi Y,X]+[Y,\phi X]\rangle_0)\\ &\quad + \tfrac{3}{4}\langle[Y,U],[Y,X]_{\mathfrak{m}}\rangle + \tfrac{1}{2}\langle[U,\phi X]+[X,\phi U],\phi^{-1}([Y,\phi Y])\rangle_0\\ &\quad - \tfrac{1}{4}\langle[U,\phi Y]+[Y,\phi U],\phi^{-1}([Y,\phi X]+[X,\phi Y])\rangle_0\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}\langle R(U,Y)Y,U\rangle &= \tfrac{1}{2}\langle[\phi U,Y]+[U,\phi Y],[Y,U]\rangle_0\\ &\quad + \tfrac{3}{4}\langle[Y,U],[Y,U]_{\mathfrak{m}}\rangle + \langle[U,\phi U],\phi^{-1}([Y,\phi Y])\rangle_0\\ &\quad - \tfrac{1}{4}\langle[U,\phi Y]+[Y,\phi U],\phi^{-1}([Y,\phi U]+[U,\phi Y])\rangle_0.\end{aligned} \tag{3.3}$$

Proof. The Chern connection of F coincides on the Levi-Civita connection of g therefore we have $R^F(U,V)W = R^g(U,V)W$, where R^F and R^g are the curvature tensors of F and g, respectively. Let $R := R^g = R^F$ be the curvature tensor of F (or g). The flag curvature is defined as follows [18]:

$$K(P,Y) = \frac{g_Y(R(U,Y)Y,U)}{g_Y(Y,Y).g_Y(U,U) - g_Y^2(Y,U)}, \tag{3.4}$$

where $g_Y(U,V) = \frac{1}{2}\frac{\partial^2}{\partial s\partial t}(F^2(Y+sU+tV))|_{s=t=0}$.

By using the definition of $g_Y(U,V)$ and some computations for F we have

$$\begin{aligned}g_Y(U,V) &= \frac{4(\sqrt{g(Y,Y)}+g(X,Y))^3}{g(Y,Y)^{5/2}}\{g(X,V)g(Y,U) - g(Y,V)g(X,U)\}\\ &\quad + \frac{2(\sqrt{g(Y,Y)}+g(X,Y))^2}{g(Y,Y)}\Bigg\{g(U,V)+g(X,U)g(X,V)\\ &\quad - \frac{g(X,Y)g(Y,V)g(Y,U)}{g(Y,Y)^{3/2}} + \frac{1}{\sqrt{g(Y,Y)}}(g(X,U)g(Y,V)\\ &\quad + g(X,Y)g(U,V) + g(X,V)g(Y,U))\Bigg\} + \frac{(\sqrt{g(Y,Y)}+g(X,Y))^4}{g(Y,Y)^3}\\ &\quad \times\{4g(Y,U)g(Y,V) - g(U,V)g(Y,Y)\} + \frac{4(\sqrt{g(Y,Y)}+g(X,Y))^2}{g(Y,Y)}\\ &\quad \times\left(\frac{g(Y,V)}{\sqrt{g(Y,Y)}} + g(X,V)\right)\left(\frac{g(Y,U)}{\sqrt{g(Y,Y)}} + g(X,U) - \frac{2g(Y,U)}{\sqrt{g(Y,Y)}}\right.\\ &\quad \left. - \frac{2g(X,Y)g(Y,U)}{g(Y,Y)}\right).\end{aligned} \tag{3.5}$$

By using (3.5) and the fact that $\{Y, U\}$ is an orthonormal basis for P with respect to g we have

$$g_Y(R(U,Y)Y,U) = (1+\langle X,Y\rangle)^2\{2\langle X,U\rangle\cdot\langle Y,R(U,Y)Y\rangle\cdot(1-2\langle X,Y\rangle) \\ +6\langle X,R(U,Y)Y\rangle\cdot\langle X,U\rangle+\langle R(U,Y)Y,U\rangle\cdot(1-\langle X,Y\rangle^2)\} \tag{3.6}$$

and

$$g_Y(Y,Y)\cdot g_Y(U,U)-g_Y^2(U,Y)=(1+\langle X,Y\rangle))^6(2\langle X,U\rangle^2-\langle X,Y\rangle^2+1). \tag{3.7}$$

Also by using Püttmann's formula (see [16] or [17]) and some computations we have

$$\langle X,R(U,Y)Y\rangle=\tfrac{1}{4}(\langle[\phi U,Y]+[U,\phi Y],[Y,X]\rangle_0+\langle[U,Y],[\phi Y,X]+[Y,\phi X]\rangle_0) \\ +\tfrac{3}{4}\langle[Y,U],[Y,X]_{\mathfrak{m}}\rangle+\tfrac{1}{2}\langle[U,\phi X]+[X,\phi U],\phi^{-1}([Y,\phi Y])\rangle_0 \\ -\tfrac{1}{4}\langle[U,\phi Y]+[Y,\phi U],\phi^{-1}([Y,\phi X]+[X,\phi Y])\rangle_0, \tag{3.8}$$

$$\langle R(U,Y)Y,Y\rangle=0 \tag{3.9}$$

and

$$\langle R(U,Y)Y,U\rangle=\tfrac{1}{2}\langle[\phi U,Y]+[U,\phi Y],[Y,X]\rangle_0+\tfrac{3}{4}\langle[Y,U],[Y,U]_{\mathfrak{m}}\rangle \\ +\langle[U,\phi U],\phi^{-1}([Y,\phi Y])\rangle_0 \\ -\tfrac{1}{4}\langle[U,\phi Y]+[Y,\phi U],\phi^{-1}([Y,\phi U]+[U,\phi Y])\rangle_0. \tag{3.10}$$

Substituting equations (3.6)–(3.10) into equation (3.4) completes the proof. □

Now we consider a special case of Riemannian homogeneous spaces which has been named naturally reductive. In this case the above formula for the flag curvature reduces to a simpler equation.

Definition 3.2 (See [13]). *A homogeneous space $M = G/H$ with a G-invariant indefinite Riemannian metric g is said to be naturally reductive if it admits an $ad(H)$-invariant decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ satisfying the condition*

$$B(X,[Z,Y]_{\mathfrak{m}})+B([Z,X]_{\mathfrak{m}},Y)=0 \qquad \text{for} \quad X,Y,Z\in\mathfrak{m},$$

where B is the bilinear form on $\mathfrak{m}$ induced by g and $[,]_{\mathfrak{m}}$ is the projection to $\mathfrak{m}$ with respect to the decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$.

Theorem 3.3. *In the previous theorem let G/H be a naturally reductive homogeneous space. Then the flag curvature of the flag (P, Y) in $T_H(G/H)$ is given by*

$$K(P,Y)=\frac{6\langle X,R(U,Y)Y\rangle.\langle X,U\rangle+\langle R(U,Y)Y,U\rangle(1-\langle X,Y\rangle^2)}{(1+\langle X,Y\rangle)^4(2\langle X,U\rangle^2-\langle X,Y\rangle^2+1)},$$

where

$$\langle X,R(U,Y)Y\rangle=\tfrac{1}{4}\langle X,[Y,[U,Y]_{\mathfrak{m}}]_{\mathfrak{m}}\rangle+\langle X,[Y,[U,Y]_{\mathfrak{h}}]\rangle \tag{3.11}$$

and

$$\langle R(U,Y)Y,U\rangle=\tfrac{1}{4}\langle U,[Y,[U,Y]_{\mathfrak{m}}]_{\mathfrak{m}}\rangle+\langle U,[Y,[U,Y]_{\mathfrak{h}}]\rangle \tag{3.12}$$

Note that $[,]_{\mathfrak{m}}$ and $[,]_{\mathfrak{h}}$ are the projections of $[,]$ to $\mathfrak{m}$ and $\mathfrak{h}$ respectively.

Proof. By using proposition 3.4 in [13] (page 202) we have

$$R(U,V)W=\tfrac{1}{4}[U,[V,W]_{\mathfrak{m}}]_{\mathfrak{m}}-\tfrac{1}{4}[V,[U,W]_{\mathfrak{m}}]_{\mathfrak{m}} \\ -\tfrac{1}{2}[[U,V]_{\mathfrak{m}},W]_{\mathfrak{m}}-[[U,V]_{\mathfrak{h}},W] \qquad \text{for} \quad U,V,W\in\mathfrak{m},$$

hence

$$R(U, Y)Y = \frac{1}{4}[Y, [U, Y]_{\mathfrak{m}}]_{\mathfrak{m}} + [Y, [U, Y]_{\mathfrak{h}}].$$

Now by substituting the last relation into the formula which is obtained in theorem 3.1, the proof is completed. □

As a special case of naturally reductive Riemannian homogeneous spaces we can consider Lie groups equipped with bi-invariant Riemannian metrics. Therefore we have the following corollary.

Corollary 3.4. *Let G be a Lie group, g be a bi-invariant Riemannian metric on G, and $\tilde{X}$ be a left invariant vector field on G such that $g(\tilde{X}, \tilde{X}) < 1$. Suppose that $F = \frac{(\alpha+\beta)^2}{\alpha}$ is the Finsler metric arising from g and $\tilde{X}$ on G such that the Chern connection of F coincides on the Levi-Civita connection of g. Then for the flag curvature of the flag $P = span\{Y, U\}$, where $\{Y, U\}$ is an orthonormal basis for P with respect to g, we have*

$$K(P, Y) = \frac{6\langle X, [Y, [U, Y]]\rangle \cdot \langle X, U\rangle + \langle U, [Y, [U, Y]]\rangle(1 - \langle X, Y\rangle^2)}{4(1 + \langle X, Y\rangle)^4(2\langle X, U\rangle^2 - \langle X, Y\rangle^2 + 1)}.$$

Proof. g is bi-invariant therefore (G, g) is naturally reductive. Now by using theorem 3.3 the proof is completed. □

Now we give some results which limit Lie groups that have a Finsler metric of type described in theorem 3.1.

Theorem 3.5. *There is no left invariant non-Riemannian (α, β)-metric of the type described in theorem 3.1 on connected Lie groups with a perfect Lie algebra, that is, a Lie algebra $\mathfrak{g}$ for which the equation $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ holds.*

Proof. If the Chern connection of F coincide on Levi-Civita connection of the left invariant Riemannian metric g then, F is of Berwald type. Therefore left invariant vector field X is parallel with respect to g and by using lemma 4.3 of [8], $g(X, [\mathfrak{g}, \mathfrak{g}]) = 0$. Since $\mathfrak{g}$ is perfect therefore X must be zero. □

Corollary 3.6. *There is no left invariant non-Riemannian (α, β)-metric of the type described in theorem 3.1 on semisimple connected Lie groups.*

Corollary 3.7. *If a Lie group G admits a left invariant non-Riemannian (α, β)-metric of the type described in theorem 3.1 then for sectional curvature of the Riemannian metric g we have*

$$K(X, u) \geqslant 0$$

for all u, where equality holds if and only if u is orthogonal to the image $[X, \mathfrak{g}]$.

Proof. Since F is of the Berwald type, X is parallel with respect to g. By using lemma 4.3 of [8], $ad(X)$ is skew-adjoint, therefore by lemma 1.2 of [14] we have $K(X, u) \geqslant 0$. □

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References

[1] Antonelli P L, Ingarden R S and Matsumoto M 1993 *The Theory of Sprays and Finsler Spaces with Applications in Physics and Biology* (Dordrecht: Kluwer)

[2] Asanov G S 1985 *Finsler Geometry, Relativity and Gauge Theories* (Dordrecht: Kluwer)

[3] Asanov G S 2005 Finsleroid space with angle and scalar product *Publ. Math. Debrecen* **67** 209–52

[4] Asanov G S 2006 Finsleroid–Finsler spaces of positive-definite and relativistic type *Rep. Math. Phys.* **58** 275–300

[5] Asanov G S 2006 Finsleroid–Finsler space and spray coefficients *Preprint* math.DG/0604526

[6] Asanov G S 2006 Finsleroid–Finsler space with Berwald and Landsberg conditions *Preprint* math.DG/0603472

[7] Bao D, Chern S S and Shen Z 2000 *An Introduction to Riemann–Finsler Geometry* (Berlin: Springer)

[8] Brown N, Finck R, Spencer M, Tapp K and Wu Z 2007 Invariant metrics with nonnegative curvature on compact Lie groups *Can. Math. Bull.* **50** 24–34

[9] Esrafilian E and Salimi Moghaddam H R 2006 Flag curvature of invariant Randers metrics on homogeneous manifolds *J. Phys. A: Math. Gen.* **39** 3319–24

[10] Esrafilian E and Salimi Moghaddam H R 2006 Induced invariant Finsler metrics on quotient groups *Balkan J. Geom. Appl.* **11** 73–9

[11] Deng S and Hou Z 2004 Invariant Finsler metrics on homogeneous manifolds *J. Phys. A: Math. Gen.* **37** 8245–53

[12] Deng S and Hou Z 2004 Invariant Randers metrics on homogeneous Riemannian manifolds *J. Phys. A: Math. Gen.* **37** 4353–60

[13] Kobayashi S and Nomizu K 1969 *Foundations of Differential Geometry* vol 2 (New york: Interscience)

[14] Milnor J 1976 Curvatures of left invariant metrics on Lie groups *Adv. Math.* **21** 293–329

[15] Nomizu K 1954 Invariant affine connections on homogeneous spaces *Am. J. Math.* **76** 33–65

[16] Püttmann T 1999 Optimal pinching constants of odd dimensional homogeneous spaces *Invent. Math.* **138** 631–84

[17] Salimi Moghaddam H R 2008 On the flag curvature of invariant Randers metrics *Mathematical Physics, Analysis and Geometry* http://dx.doi.org/10.1007/s11040-008-9037-8

[18] Shen Z 2001 *Lectures on Finsler Geometry* (Singapore: World Scientific)

[19] Shen Z and Yildirim G C 2008 On a class of projectively flat metrics with constant flag curvature *Can. J. Math.* **60** 443–56